

# AP CALCULUS AB & BC

## Course Description

Calculus is an advanced mathematics course that uses meaningful problems and appropriate technology to develop concepts and applications related to continuity and discontinuity of functions and differentiation, and integration. The advanced placement courses will use a curriculum based upon the curriculum of the state of Tennessee and the guidelines of the College Board. Items for the BC course will be denoted with an asterisk.

Section references appear in brackets after each student performance indicator. They are for our adopted textbook: Calculus: Graphical, Numerical, Algebraic by Finney, et. al.

## Standard 1.0: Functions, Graphs, and limits

### Learning Expectations:

Students will:

- 1.1 Analyze the graphs of functions and relations;
- 1.2 Evaluate the limits of functions (including one-sided limits);
- 1.3 Analyze asymptotic and unbounded behavior;
- 1.4 Understand continuity as a property of functions;
- 1.5 \* Analyze parametric, polar, and vector functions.

### Student Performance Indicators:

- 1.1.a Graph functions on a graphing calculator using the appropriate windows. [1.1 – 1.6]
- 1.1.b Recognize functions by type: linear, quadratic, polynomial, rational, exponential, logarithmic, power, roots, absolute value, [1.1 – 1.6]
- 1.1.c \* Recognize functions by type: parametric, polar, and vector. [10.1, 10.3, 10.5]
- 1.1.d Predict and explain the observed local and global behavior of a function. [4.1 – 4.3]
  
- 1.2.a Demonstrate an intuitive understanding of the limiting process. [2.1]
- 1.2.b Calculate limits using algebra. [2.1]
- 1.2.c Estimate limits from graphs or tables of data. [2.1]
  
- 1.3.a Demonstrate an understanding of asymptotes in terms of graphical behavior. [2.2]
- 1.3.b Describe asymptotic behavior in terms of limits involving infinity. [2.2]
- 1.3.c Compare relative magnitudes of functions and their rates of change. (For example, contrasting exponential growth, polynomial growth, and logarithmic growth.) [8.2]
  
- 1.4.a Develop an intuitive understanding of continuity. (Close values of the domain lead to close values of the range.) [3.3]
- 1.4.b Develop and understanding of continuity in terms of limits. [2.4]
- 1.4.c Develop a geometric understanding of graphs of continuous functions. (Intermediate Value Theorem and Extreme Value Theorem.) [2.3 and 4.1]
  
- 1.5.a \* Analyze and graph curves in parametric form. [10.1]
- 1.5.b \* Analyze and graph curves in polar form. [10.5]
- 1.5.c \* Analyze and graph curves in vector form. [10.3]

## Standard 2.0: Derivatives

### Learning Expectations:

Students will:

- 2.1 Develop the concepts of the derivative;
- 2.2 Have an understanding of the derivative at a point;
- 2.3 Investigate the derivative as a function;
- 2.4 Explore second derivatives;
- 2.5 Apply derivatives;
- 2.6 Compute derivatives.

### Student Performance Indicators:

- 2.1.a Compute derivatives graphically, numerically, and analytically. [3.1 – 3.3]
- 2.1.b Interpret a derivative as an instantaneous rate of change. [2.4]
- 2.1.c Define the derivative as the limit of the difference quotient. [3.1]
- 2.1.d Understand the relationship between differentiability and continuity. [3.2]
  
- 2.2.a Determine the slope of a curve at a point. Examples to emphasize include points at which there are vertical tangents and points at which there are no tangents. [3.1 – 3.2]
- 2.2.b Determine tangent lines to a curve at a point and local linear approximations. [3.1, 4.5]
- 2.2.c Compute the instantaneous rate of change as the limit of the average rate of change. [2.8, 3.1, 3.4]
- 2.2.d Approximate the rate of change from graphs and tables of values. [3.1]
  
- 2.3.a Determine the corresponding characteristics of the graphs of  $f$  and  $f'$ . [3.1, 3.4]
- 2.3.b Determine the relationship between the increasing and decreasing behavior of  $f$  and the sign of  $f'$ . [4.2]
- 2.3.c Investigate the Mean Value Theorem and its geometric consequences. [4.2]
- 2.3.d Explore equations involving derivatives. Verbal descriptions are to be translated into equations involving derivatives and vice versa. [3.4, 4.4, 4.6]
  
- 2.4.a Explore the corresponding characteristics of the graphs of  $f$ ,  $f'$ , and  $f''$ . [4.3]
- 2.4.b Determine the relationship between the concavity of  $f$  and the sign of  $f''$ . [4.3]
- 2.4.c Find the points of inflection as places where the concavity changes. [4.3]
  
- 2.5.a Analyze curves, including the notions of monotonicity and concavity. [4.3]
- 2.5.b \* Analysis of planar curves given in parametric form, polar form, and vector form, including velocity and acceleration vectors. [10.3, 10.6]
- 2.5.c Perform optimization for both absolute (global) and relative (local) extrema. [4.4]
- 2.5.d Model rates of change, including related rates problems. [4.6]
- 2.5.e Use implicit differentiation to find the derivative of an inverse function. [3.8]

- 2.5.f Interpret the derivative as a rate of change in varied applied contexts, including velocity, speed, and acceleration. [3.4]
- 2.5.g Investigate the geometric interpretation of differential equations via slope fields and the relationship between slope fields and solution curves for differential equations. [6.1]
- 2.5.h \* Compute the numerical solution of differential equations using Euler's method. [6.6]
- 2.5.i \* Apply L'Hôpital's Rule, including its use in determining limits and convergence of improper integrals and series. [8.1]
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- 2.6.a Attain knowledge of derivatives of basic functions, including power, exponential, logarithmic, trigonometric, and inverse trigonometric functions. [3.3, 3.5, 3.8, 3.9]
- 2.6.b Apply the basic rules for the derivative of sums, products, and quotients of functions. [3.3]
- 2.6.c Apply the chain rule and implicit differentiation. [3.6, 3.7]
- 2.6.d \* Evaluate the derivatives of parametric, polar, and vector functions. [10.1, 10.3, 10.6]

## Standard 3.0: Integrals

### Learning Expectations:

Students will:

- 3.1 Discover the interpretations and the properties of the definite integral;
- 3.2 Apply integrals;
- 3.3 Discover the Fundamental Theorem of Calculus;
- 3.4 Explore techniques of antidifferentiation;
- 3.5 Apply antiderivatives;
- 3.6 Compute numerical approximations to definite integrals.

### Student Performance Indicators:

- 3.1.a Compute Riemann sums using left, right, and midpoint evaluation points. [5.1]
- 3.1.b Determine the definite integral as a limit of Riemann sums over equal subdivisions. [5.2]
- 3.1.c Determine the definite integral of the rate of change of a quantity over an interval interpreted as the change of the quantity over the interval:  $\int_a^b f'(x)dx = f(b) - f(a)$  [7.1]
- 3.1.d Explore the basic properties of definite integrals. (Examples include additivity and linearity.) [6.1]
  
- 3.2.a \* Use appropriate integrals in a variety of applications to model physical, biological, or economic situations. [7.5]
- 3.2.b Use knowledge and techniques for solving applications and adapt this knowledge to solve similar application problems. [7.1 – 7.5]
- 3.2.c Use the integral of a rate of change to give accumulated change. [7.1 – 7.5]
- 3.2.d Represent a Riemann sum as a definite integral. [5.4]
- 3.2.e Find the area of a region. [5.2, 5.3, 7.2]
- 3.2.f \* Find the area of a region bounded by polar curves. [10.6]
- 3.2.g Find the volume of a solid with known cross sections. [7.3]
- 3.2.h Find the average value of a function. [5.3]
- 3.2.i Find the distance traveled by a particle along a line. [7.1]
- 3.2.j \* Find the length of a curve. [7.4]
- 3.2.k \* Find the length of a curve in parametric form and polar form. [10.1, 10.6]
  
- 3.3.a Use the Fundamental Theorem to evaluate definite integrals. [5.3, 5.4]
- 3.3.b Use the Fundamental Theorem to represent a particular antiderivatives, and the analytic and graphical analysis of functions so defined. [5.3, 5.4]
  
- 3.4.a Explore how antiderivatives follow directly from derivatives of basic functions. [6.1]

3.4.b \* Determine antiderivatives by substitution of variables (including change of limits for definite integrals), parts, and simple partial fractions (nonrepeating linear factors only). [6.2, 6.3, 8.4]

3.4.c \* Determine improper integrals (as limits of definite integrals). [8.3]

3.5.a Find specific antiderivatives using initial conditions, including applications to motion along a line. [6.1, 7.1]

3.5.b Solve separable differential equations and using them in modeling. In particular, studying the equation  $y' = ky$  and exponential growth. [6.1, 6.4]

3.5.c \* Solve logistic differential equations and using them in modeling. [6.5]

3.6.a Use Riemann and trapezoidal sums to approximate definite integrals of functions represented algebraically, graphically, and by tables of values. [5.1, 5.2, 5.5]

## Standard 4.0: \* Polynomial Approximations and Series

### Learning Expectations:

Students will:

- 4.1 Develop the concepts of series;
- 4.2 Explore series of constants.

### Student Performance Indicators:

- 4.1.a. Define a series as a sequence of partial sums.
- 4.1.b. Define convergence in terms of the limit of the sequence of partial sums.
- 4.1.c. Use technology to explore convergence or divergence.
  
- 4.2.a. Use motivating examples such as decimal expansion. [9.1]
- 4.2.b. Investigate geometric series with applications. [9.1]
- 4.2.c. Define the harmonic series. [9.5]
- 4.2.d. Find the error bound of an alternating series. [9.5]
- 4.2.e. Relate improper integrals to series. [9.5]
- 4.2.f. Use the integral test to determine the convergence of p-series. [9.5]
- 4.2.g. Use the ratio to determine convergence or divergence. [9.4]
- 4.2.h. Compare series to test for convergence or divergence. [9.4, 9.5]

## Section Summary And Pacing Guide For Calculus AB

To cover the course outline for Calculus AB, a teacher must cover:

- Chapter 2: Sections 1, 2, 3, and 4
- Chapter 3: Sections 1, 2, 3, 4, 5, 6, 7, 8, and 9
- Chapter 4: Sections 1, 2, 3, 4, and 6
- Chapter 5: Sections 1, 2, 3, 4, and 5
- Chapter 6: Sections 1, 2, and 4
- Chapter 7: Sections 1, 2, and 3

What follows is a suggested pacing guide that attempts to complete the course content for AP Calculus AB in one semester - specifically, the spring semester when there are fewer days due to the placement of the exam day.

Day	Section	Topic
1	Quiz	Quiz: Sections 1.1 – 1.6. Skip section 1.4
2	2.1	Rates of Change and Limits
3	2.2	Limits Involving Infinity
4	Quiz	Quiz: Limits
5	2.3	Continuity
6	2.4	Rates of Change and Tangent Lines
7	Quiz	Quiz: Chapter 2
8	3.1	Derivative of a Function
9		Derivative of a Function (continued)
10	3.2	Differentiability
11		Review
12	Test	Test: Chapter 2 and Sections 3.1 – 3.2
13	3.3	Rules of Differentiation
14	3.4	Velocity and Other Rates of Change
15		Velocity and Other Rates of Change (continued)
16	3.5	Derivatives of Trigonometric Functions
17		Derivatives of Trigonometric Functions (continued)
18	3.6	Chain Rule
19		Chain Rule (continued)
20		Review
21	Quiz	Quiz: 3.4 – 3.6
22	3.7	Implicit Differentiation
23		Implicit Differentiation (continued)
24	3.8	Derivatives of Inverse Trigonometric Functions
25	3.9	Derivatives of Exponential and Logarithmic Functions
26		Derivatives of Exponential and Logarithmic Functions (continued)
27		Concept Connections
28		Review
29	Test	Test: Chapter 3
30	4.1	Extreme Values of Functions, Optional - Section 4.5
31	4.2	Mean Value Theorem

32	4.3	Connecting Derivatives to the Graph of a Function
33		Connecting Derivatives to the Graph of a Function (continued)
34		Concept Connections
35	Quiz	Quiz: 4.1 – 4.3
36	4.4	Modeling and Optimization
37		Modeling and Optimization (continued)
38	4.6	Related Rates
39		Related Rates (continued)
40		Related Rates (continued)
41		Review
42	Test	Test: Chapter 4
43	5.1	Estimating with Finite Sums
44	5.2	Definite Integrals
45	5.3	Definite Integrals and Antiderivatives
46		Concept Connections
47	5.4	Fundamental Theorem of Calculus
48		Concept Connections
49	5.5	Trapezoidal Rule (Skip Simpson's Rule)
50		Review
51	Test	Test: Chapter 5
52	6.1	Antiderivatives and Slope Fields
53	6.2	Integration by Substitution
54		Integration by Substitution (continued)
55	6.4	Exponential Growth and Decay
56		Exponential Growth and Decay (continued)
57	Quiz	Quiz: Chapter 6
58	7.1	Integral as Net Change
59	7.2	Areas in a Plane
60	7.3	Volumes
61		Volumes (continued)
62		Volumes (continued)
63	Quiz	Quiz 7.1-7.3
64		Review
65		Review
66		Review
67		Review
68		Review
69		Review
70		Review
71		Review
72		Review
73		Review
74		AP Exam/Semester Exam

## Section Summary And Pacing Guide For Calculus BC

To cover the course outline for Calculus BC, a teacher must cover:

- Chapter 1: Section 4
- Chapter 2: Sections 1, 2, 3, and 4
- Chapter 3: Sections 1, 2, 3, 4, 5, 6, 7, 8, and 9
- Chapter 4: Sections 1, 2, 3, 4, and 6
- Chapter 5: Sections 1, 2, 3, 4, and 5
- Chapter 6: Sections 1, 2, 3, 4, 5, and 6
- Chapter 7: Sections 1, 2, 3, 4, and 5
- Chapter 8: Sections 1, 2, 3, and 4
- Chapter 9: Sections 1, 2, 3, 4, and 5
- Chapter 10: Sections 1, 2, 3, 5, and 6

What follows is a suggested pacing guide that attempts to complete the course content for AP Calculus BC in two semesters.

Day	Section	Topic
1	1.4	Parametric Equations
2	Quiz	Quiz: Sections 1.1 – 1.6.
3	2.1	Rates of Change and Limits
4	2.2	Limits Involving Infinity
5	Quiz	Quiz: Limits
6	2.3	Continuity
7	2.4	Rates of Change and Tangent Lines
8	Quiz	Quiz: Chapter 2
9	3.1	Derivative of a Function
10		Derivative of a Function (continued)
11	3.2	Differentiability
12		Review
13	Test	Test: Chapter 2 and Sections 3.1 – 3.2
14	3.3	Rules of Differentiation
15	3.4	Velocity and Other Rates of Change
16		Velocity and Other Rates of Change (continued)
17	3.5	Derivatives of Trigonometric Functions
18		Derivatives of Trigonometric Functions (continued)
19	3.6	Chain Rule
20		Chain Rule (continued)
21		Review
22	Quiz	Quiz: 3.4 – 3.6
23	3.7	Implicit Differentiation
24		Implicit Differentiation (continued)
25	3.8	Derivatives of Inverse Trigonometric Functions
26	3.9	Derivatives of Exponential and Logarithmic Functions
27		Derivatives of Exponential and Logarithmic Functions (continued)

28		Concept Connections
29		Review
30	Test	Test: Chapter 3
31	4.1	Extreme Values of Functions
32	4.2	Mean Value Theorem
33	4.3	Connecting Derivatives to the Graph of a Function
34		Connecting Derivatives to the Graph of a Function (continued)
35		Concept Connections
36	Quiz	Quiz: 4.1 – 4.3
37	4.4	Modeling and Optimization
38		Modeling and Optimization (continued)
39	4.5	Linearization and Newton’s Method (optional)
40	4.6	Related Rates
41		Related Rates (continued)
42		Related Rates (continued)
43		Review
44	Test	Test: Chapter 4
45	5.1	Estimating with Finite Sums
46	5.2	Definite Integrals
47	5.3	Definite Integrals and Antiderivatives
48		Concept Connections
49	5.4	Fundamental Theorem of Calculus
50		Concept Connections
51	5.5	Trapezoidal Rule (Skip Simpson’s Rule)
52		Review
53	Test	Test: Chapter 5
54	6.1	Antiderivatives and Slope Fields
55	6.2	Integration by Substitution
56		Integration by Substitution (continued)
57	6.3	Integration by Parts
58		Integration by Parts (continued)
59	Quiz	Quiz: 6.1 – 6.3
60	6.4	Exponential Growth and Decay
61		Exponential Growth and Decay (continued)
62	8.4	Partial Fractions (Introduce non-repeated linear factors)
63	6.5	Population Growth
64		Population Growth (continued)
65	6.6	Numerical Methods (Euler’s Method only)
66		Review
67	Test	Test: Chapter 6
68	7.1	Integral as Net Change
69	7.2	Areas in a Plane
70	7.3	Volumes
71		Volumes (continued)
72		Volumes (continued)

73	Quiz	Quiz 7.1-7.3
74	7.4	Lengths of Curves
75	7.5	Applications from Science and Statistics (work)
76		Review
77	Test	Test: Chapter 7
78		Review
79		Review
80		Review
81		Review
82		Review
83		Review
84		Review
85	Test	Semester Exam
86	8.1	L'Hôpital's Rule
87		L'Hôpital's Rule (continued)
88	8.2	Relative Rates of Growth (Skip Oh-notation)
89		Relative Rates of Growth (continued)
90	Quiz	Quiz: 8.1 – 8.2
91	8.3	Improper Integrals
92		Improper Integrals (continued)
93	8.4	Partial Fractions (Only linear non-repeating is required)
94		Partial Fractions (Non linear and repeating – optional)
95		Partial Fractions (Trigonometric substitution – optional)
96		Review
97	Test	Test: Chapter 8
98	9.1	Power Series
99		Power Series (continued)
100		Power Series (continued)
101	9.2	Taylor Series
102		Taylor Series (continued)
103		Taylor Series (continued)
104	9.3	Taylor's Theorem
105		Taylor's Theorem (continued)
106		Taylor's Theorem (continued)
107	9.4	Radius of Convergence
108		Radius of Convergence (continued)
109		Radius of Convergence (continued)
110	9.5	Testing Convergence at Endpoints
111		Testing Convergence at Endpoints (continued)
112		Testing Convergence at Endpoints (continued)
113		Review
114	Test	Test: Chapter 9
115	1.4	Parametric Equations
116		Parametric Equations (continued)
117	10.1	Parametric Functions

118		Parametric Functions (continued)
119	10.2	Vectors in the Plane
120		Vectors in the Plane (continued)
121	10.3	Vector-valued Functions
122		Vector-valued Functions (continued)
123		Vector-valued Functions (continued)
124	10.4	Modeling Projectile Motion (optional)
125	10.5	Polar Coordinates and Polar Graphs
126		Polar Coordinates and Polar Graphs (continued)
127		Polar Coordinates and Polar Graphs (continued)
128	10.6	Calculus of Polar Curves
129		Calculus of Polar Curves (continued)
130		Calculus of Polar Curves (continued)
131		Review
132	Test	Test: Chapter 10
133	Review	

Once chapter 10 is completed, the rest of the semester is devoted to review for the advanced placement exam. The date for the exam is usually during the first week of May. It should be noted that some teachers mix their exam review days throughout the term while others save it for the end of the term as noted in the outline above.

## AP Calculus Resources

Among the materials to help students prepare for the A.P. Exam, it is recommended that the teacher have access to previous exams. The free-response portions are available after the exam each year. Solutions to the previous few years can be viewed at the advanced placement website (<http://apcentral.collegeboard.com>). The multiple-choice portion of the exam can be reviewed through released exams that can be purchased from the College Board. They release certain exams at irregular intervals. In addition to these actual exams, there are many A.P. review texts available for purchase. As is typical of standardized test reviews, many of these texts tend to be more difficult than the actual exam.

We encourage teachers new to A.P. Calculus to spend time and gather resources from those teachers who have taught the course previously. What follows is an overview of what students should know before taking the exam. Teachers might want to provide their students with a copy.

# STUFF YOU SHOULD KNOW BEFORE TAKING THE CALCULUS ADVANCED PLACEMENT EXAM

## 1. ALL OF THE BASIC DIFFERENTIATION AND INTEGRATION RULES.

### DIFFERENTIATION

- $\frac{d}{dx} [cu] = c u'$  \* you can factor a constant out
- $\frac{d}{dx} [u \pm v] = u' \pm v'$
- $\frac{d}{dx} [uv] = uv' + vu'$  \*product rule
- $\frac{d}{dx} \left[ \frac{u}{v} \right] = \frac{vu' - uv'}{v^2}$  \* quotient rule: ho d[hi] - hi d[ho] over ho ho
- $\frac{d}{dx} [c] = 0$
- $\frac{d}{dx} [u^n] = n u^{(n-1)} u'$  \*power rule
- $\frac{d}{dx} [x] = 1$
- $\frac{d}{dx} [|u|] = \frac{u}{|u|} u'$  \*rarely used
- $\frac{d}{dx} [\ln u] = \frac{u'}{u}$
- $\frac{d}{dx} [e^u] = e^u u'$
- $\frac{d}{dx} [\log_a u] = \frac{u'}{(\ln a)u}$  \*works like ln, but you divide by (ln a)
- $\frac{d}{dx} [a^u] = a^u (\ln a) u'$  \*works like e, but you mult by (ln a)
- $\frac{d}{dx} [\sin u] = (\cos u) u'$
- $\frac{d}{dx} [\cos u] = -(\sin u) u'$
- $\frac{d}{dx} [\tan u] = (\sec^2 u) u'$
- $\frac{d}{dx} [\cot u] = -(\csc^2 u) u'$

17.  $\frac{d}{dx} [\sec u] = (\sec u \tan u) u'$
18.  $\frac{d}{dx} [\csc u] = -(\csc u \cot u) u'$  \*every other trig is negative, but similar
19.  $\frac{d}{dx} [\arcsin u] = \frac{u'}{\sqrt{1-u^2}}$
20.  $\frac{d}{dx} [\arccos u] = \frac{-u'}{\sqrt{1-u^2}}$
21.  $\frac{d}{dx} [\arctan u] = \frac{u'}{u^2+1}$
22.  $\frac{d}{dx} [\operatorname{arccot} u] = \frac{-u'}{u^2+1}$
23.  $\frac{d}{dx} [\operatorname{arcsec} u] = \frac{u'}{|u|\sqrt{u^2-1}}$
24.  $\frac{d}{dx} [\operatorname{arccsc} u] = \frac{-u'}{|u|\sqrt{u^2-1}}$  \*every other inverse trig is the opposite

## INTEGRATION

1.  $\int k \cdot f(u) \, du = k \cdot \int f(u) \, du$
2.  $\int [f(u) \pm g(u)] \, du = \int f(u) \, du \pm \int g(u) \, du$
3.  $\int du = u + c$
4.  $\int u^n \, du = \frac{u^{n+1}}{n+1} + c$
5.  $\int \frac{du}{u} = \ln |u| + c$
6.  $\int e^u \, du = e^u + c$
7.  $\int a^u \, du = \frac{1}{\ln a} a^u + c$
8.  $\int \sin u \, du = -\cos u + c$
9.  $\int \cos u \, du = \sin u + c$
10.  $\int \tan u \, du = -\ln |\cos u| + c$
11.  $\int \cot u \, du = \ln |\sin u| + c$
12.  $\int \sec u \, du = \ln |\sec u + \tan u| + c$
13.  $\int \csc u \, du = -\ln |\csc u + \cot u| + c$

$$14. \int \sec^2 u \, du = \tan u + c$$

$$15. \int \csc^2 u \, du = -\cot u + c$$

$$16. \int \sec u * \tan u \, du = \sec u + c$$

$$17. \int \csc u * \cot u \, du = -\csc u + c$$

$$18. \int \frac{du}{\sqrt{a^2 - u^2}} = \arcsin \frac{u}{a} + c$$

$$19. \int \frac{du}{a^2 + u^2} = \frac{1}{a} \arctan \frac{u}{a} + c$$

$$20. \int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \operatorname{arcsec} \frac{|u|}{a} + c$$

## 2. LIMITS AND CONTINUITY:

A) Two special limits:  $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$ ,    B)  $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$

B) In general, if you can plug a limit it in, then plug it in

C) If you cannot just plug a limit in, then you can investigate the limit in a variety of ways

I) Numerically - make a table of values

II) Analytically - remove any discontinuity and then plug in

III) Graphically - where is the value of y heading for your particular x?

D) Sometimes limits do not exist:

I) diverge to infinity

II) oscillating functions

III) when the limit on the left is not equal to the limit on the right

E) Limits at infinity (also horizontal asymptotes).

Check the degree on the top (T) and the bottom (B):

I) if  $T > B$ , then it diverges to infinity

II) if  $T = B$ , then it is the ratio of the leading coefficients

III) if  $T < B$ , then it is zero

Note that it is possible that the left and right end behavior may differ. As an example:

$$\lim_{x \rightarrow -\infty} \frac{3x - 2}{\sqrt{4x^2 + 5}} = -\frac{3}{2} \quad \text{while} \quad \lim_{x \rightarrow \infty} \frac{3x - 2}{\sqrt{4x^2 + 5}} = \frac{3}{2}$$

F) A point is continuous if the limit on the left is the limit on the right and these equal the value of the function. An interval is continuous if every point on the interval is continuous.

G) A named theorem: The Intermediate Value Theorem. If  $f(x)$  is a continuous function on  $[a, b]$ , then for any  $y_0$  between  $f(a)$  and  $f(b)$  there exists a "c" within  $[a, b]$  such that  $f(c) = y_0$ .

### 3. THE DEFINITION OF THE DERIVATIVE

A)  $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$  if the limit exists. This is the limit of the difference quotient.

B) You should also be familiar with the alternate definition of the derivative at a point "a":

$$f'(a) = \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

### 4. DERIVATIVE THOUGHTS

A) If a function is differentiable on an interval, then it is continuous as well.

B) The derivative evaluated at a point is the slope of the line tangent to the function at that point.

C) The derivative of the position function is the velocity function and the derivative of the velocity function is the acceleration function.

D) The graph  $f'(x)$  will have zeros at the relative extrema of a differentiable function  $f(x)$ .  $f'(x)$  will be above the x-axis where  $f(x)$  is rising and below the x-axis where  $f(x)$  is falling.

E) For a piece-wise defined function, if the derivative is the same on both sides then it is considered "smooth" and the derivative exists.

F) The chain rule for the derivative of composite function  $y = f(g(x))$  is

$$\frac{d}{dx} y = f'(g(x)) * g'(x). \text{ In another notation, if } u = g(x), \text{ then } y = f(u), \text{ so the}$$

$$\text{derivative w.r.t. } x \text{ is: } \frac{dy}{dx} = \frac{dy}{du} \frac{du}{dx}$$

### 5. CRITICAL NUMBERS AND THE FIRST AND SECOND DERIVATIVE

Critical numbers exist where the first derivative = 0 or does not exist for points that are within the domain (i.e. points that exist). The first derivative is used to find intervals where the function is increasing or decreasing [include the endpoints if they exist]. The second derivative indicates intervals where the function is concave up or down (always use open intervals). Points of inflection exist where the intervals of concavity change on either side of the point.

### 6. RELATIVE AND ABSOLUTE MAX AND MIN

A) Relative extrema are on an open interval. A relative maximum exists where the interval on the left of a critical point is increasing while the interval on the right is decreasing (This is the first derivative test for relative extrema). A relative maximum

also occurs where a critical point exists in an interval that is concave down (This is the second derivative test for relative extrema). \*Mins are just the opposite\*.

B) Absolute maximums and minimums also occur either at the relative extrema or at closed endpoints. This is another named theorem: The Extreme Value Theorem. If  $f$  is continuous on a closed interval  $[a, b]$ , then  $f$  has both a maximum value and a minimum value on the interval.

## 7. DERIVATIVE APPLICATIONS

A) Another named theorem: The Mean Value Theorem. There is a point,  $c$ , in the closed interval  $[a, b]$  on differentiable function  $f(x)$  such that  $f'(c) = \frac{f(b) - f(a)}{b - a}$ .

B) The M.V.T enables us to find the point where the instantaneous rate of change is equal to the average rate of change. It is where the slope of the tangent line is equal to the slope of the secant line.

C) A technique for optimization (max/min) problems:

- Write the equation of what is to be maximized or minimized.
- Substitute until the right side is in terms of one variable, (the left side will be the variable you will want to optimize).
- Take the derivative, set it equal to zero, and solve.

D) A technique for related rates problems

- Find an equation relating the variables you are interested in
- Get the equation in terms of two variables (you may have to use a secondary equation)
- Take the derivative (implicit) of both sides with respect to  $t$
- One of the derivatives will be given, plug it in
- Solve for the other derivative
- Answer the question - you may have to work to get the value of the variable you are going to plug in.

## 8. CURVE SKETCHING CONSIDERATIONS

A) Discontinuities - removable (holes), non-removable (vertical asymptotes)

B) Limits at infinity - possible horizontal asymptotes

C)  $x$  and  $y$  intercepts - set the other = 0 and solve

D) Relative max and mins - occur at critical points. Use 1st or 2nd derivative test.

E) Concavity - up when  $f''(x) > 0$ , down when  $f''(x) < 0$

F) One sided limits at the vertical asymptotes - will be  $+$  or  $-\infty$ , graph or plug in

G) Translations and dilations and reflections - see #9 below.

9. Manipulation of functions (from Pre-Calculus): Consider  $y = f(x)$

A)  $y = f(x - h)$  shifts  $h$  units horizontally

B)  $y = f(x) + k$  shifts  $k$  units vertically

C)  $y = A \cdot f(x)$  stretches vertically by a factor of  $A$

- D)  $y = f(Bx)$  shrinks horizontally by a factor of B
- E)  $y = -f(x)$  reflects through the x-axis
- F)  $y = f(-x)$  reflects through the y-axis
- G)  $y = |f(x)|$  flips those portions below the x-axis up
- H)  $y = f(|x|)$  keeps the right side and flips it through the y-axis

## 10. THE DEFINITE INTEGRAL

A) Estimating the area under a curve by means of LRAM, RRAM, MRAM, and TRAP

The trapezoid rule is  $TRAP = \frac{1}{2}(y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n) * \Delta X$  where

$$\Delta X = \frac{b-a}{n}$$

B) If  $f$  is above the x-axis on  $[a, b]$ , then the area under the curve is  $\int_a^b f(x)dx$

## 11. THE FUNDAMENTAL THEOREM OF CALCULUS

A) The definite integral is the difference of the boundaries evaluated at the antiderivative:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

B) Another version of the F.T.C:

I) If the upper limit of integration is just plain  $x$ :  $D_x \left[ \int_a^x f(t)dt \right] = f(x)$

II) In general with the chain rule:

$$D_x \left[ \int_{g(x)}^{h(x)} f(t)dt \right] = f(h(x)) * h'(x) - f(g(x)) * g'(x)$$

## 12. APPLICATIONS OF INTEGRATION

A) Area between two curves  $\int_{left}^{right} (top - bottom)dx$

B) The average value of a function:  $\frac{1}{b-a} \int_a^b f(x)dx$

C) Volumes of Revolution

I) Disks around the x-axis (or other horizontal axis,  $y = h$ )

$$v = \pi \int_a^b [(outside - axis)^2 - (inside - axis)^2] dx$$

II) While this is no longer required for A.P., Shells around the y-axis (or other vertical axis,  $x = k$ )

$$v = 2\pi \int_a^b p(x) * (topcurve - bottomcurve) dx$$

$$\text{where } p(x) = \begin{cases} x - k & \text{if the axis is on the left of the region} \\ k - x & \text{if the axis is on the right of the region} \end{cases}$$

#### D) Volumes by Cross Section

I) If a shape has a known cross section whose area is a function of  $x$ ,  $A(x)$ , you can integrate over its length to get the volume:  $V = \int_{left}^{right} A(x) dx$

II) If the cross-sections are between the curve and the x-axis:

a) Squares:  $V = \int_{left}^{right} [f(x)]^2 dx$

b) Equilateral Triangles:  $V = \frac{\sqrt{3}}{4} \int_{left}^{right} [f(x)]^2 dx$

c) Semi-Circles:  $V = \frac{\pi}{8} \int_{left}^{right} [f(x)]^2 dx$

#### E) Rectilinear motion (motion along a line):

I) Total distance =  $\int_{beginningtime}^{endtime} |v(t)| dt$

II) Displacement =  $\int_{beginningtime}^{endtime} v(t) dt$

G) An accumulator function is a function defined as the integral from a constant to  $x$ .

H)  $\int_{beginningtime}^{endtime} RateOfSomething * dt = \text{Net change in the something.}$

### 13. SOLVING DIFFERENTIAL EQUATIONS & GROWTH AND DECAY PROBLEMS

A) In general, separate your variables and integrate both sides.

B) This is where the natural log often creeps in. If the rate of change is proportional to the amount present, ( $dy/dt = k \cdot y$ ), then you can go right to  $y = ce^{kt}$ . You could also prove this by separating variables. For Newton's Law of cooling, [ $dy/dt = k(y - q)$  where  $q$  is the constant temperature],  $y = ce^{kt} + q$

C) A slope field can be generated for a first order differential equation. The solution of a differential equation is a family of graphs that will pass "gently" through the slope field. If you have a specific point you can solve for the constant of integration and find the exact function.

D) We use Euler's method to numerically solve a differential equation. Given an initial point  $(x_0, y_0)$ :  $x_{n+1} = x_n + dx$ ,  $y_{n+1} = y_n + y'_n \cdot dx$

E) To sketch a function  $f(x)$  when given  $f'(x)$ : The zeros of  $f'(x)$  that pass through the  $x$ -axis (as opposed to bouncing off) will be the relative extrema of  $f(x)$ , the relative extrema of  $f'(x)$  will be points of inflection of  $f(x)$ . Where  $f'(x)$  is falling  $f(x)$  is concave down. Where  $f'(x)$  is rising  $f(x)$  is concave up.

#### 14. OTHER FUNCTIONS FOR THE BC STUDENT AND ASSOCIATED FORMULAS

A) Additional formulas for functions:

I) The arc length of a curve:  $L = \int_{\text{left}}^{\text{right}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$

II) Integration By Parts:  $\int u dv = uv - \int v du$

III) The logistic differential equation  $\frac{dP}{dt} = kP\left(1 - \frac{P}{M}\right)$ , has the solution

$$P = \frac{M}{1 + Ae^{-kt}}$$

B) Parametric functions

I)  $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$

II)  $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt$

C) Vector

I) Definitions for particle movement.

a)  $\mathbf{v}(t) = \frac{d\mathbf{r}}{dt}$  is the particle's velocity vector.

b)  $|\mathbf{v}(t)|$ , the magnitude of  $\mathbf{v}$ , is the particle's speed.

c)  $\mathbf{a}(t) = \frac{d\bar{v}}{dt} = \frac{d^2\bar{r}}{dt^2}$  is the particles acceleration vector.

d)  $\frac{\mathbf{v}}{|\mathbf{v}|}$ , a unit vector, is the direction of motion

II) If the initial position is  $\mathbf{r}(0) = h\mathbf{i} + k\mathbf{j}$ , then

$$\mathbf{r}(t) = (v_0 \cos \theta t + h)\mathbf{i} + \left(\frac{-1}{2}gt^2 + v_0 \sin \theta t + k\right)\mathbf{j}$$

D) Polar

$$\text{I) } \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{f'(\theta)\sin(\theta) + f(\theta)\cos(\theta)}{f'(\theta)\cos(\theta) - f(\theta)\sin(\theta)}$$

$$\text{II) } A = \int_{\alpha}^{\beta} \frac{1}{2}r^2 d\theta$$

$$\text{III) } L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$$

## 15. INFINITE POSSIBILITIES FOR THE BC STUDENT

A) L'Hopital's Rule; if  $f(a) = g(a) = 0$  and  $f$  and  $g$  are differentiable on an open interval  $I$  containing  $a$ , and if  $g'(x) \neq 0$  on  $I$  as long as  $x \neq a$ , then:

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

B) L'Hopital's Rule works for  $a = \infty$  (as  $x$  goes to infinity) and if

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$$

C) We take care of improper integrals as follows:

I) If one of the limits of integration is infinite, we assign it a letter like "a"

$$\text{and then take } \lim_{a \rightarrow \infty}$$

II) If both limits of integration are infinite, break the integral into two pieces each having only one infinite limit of integration.

III) If one of the limits of integration is a vertical asymptote, we assign it a letter and then take the limit as the variable approaches that letter from the appropriate side.

IV) If a vertical asymptote exists within the interval, break the integral into two pieces each having only one limit of integration needing to be approached by a certain side.

D)  $\int_1^{\infty} \frac{1}{x^p} dx$  will converge if  $p > 1$  and diverge otherwise.

E)  $\int_0^1 \frac{1}{x^p} dx$  will diverge if  $p \geq 1$  and will converge if  $p < 1$ .

F) The Limit Comparison Test compares a function to a known function: If the positive functions  $f$  and  $g$  are continuous on  $[a, \infty)$  and if  $\lim_{x \rightarrow \infty} \frac{f(x)}{g(x)} = L$ ,

$0 < L < \infty$ , then

$\int_a^{\infty} f(x) dx$  and  $\int_a^{\infty} g(x) dx$  either both converge or both diverge.

## 16. INFINITE SERIES FOR THE BC STUDENT

A) A Taylor series centered at  $x = a$  is of the form:

$$\frac{f(a)}{0!} + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!}(x-a)^4 + \dots$$

B) A Maclaurin Series is centered at  $x = 0$ . Here are a few to memorize:

1)  $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$

2)  $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

3)  $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$

4)  $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$

C) A Taylor polynomial centered at  $x = a$  is of the form

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x-a)^k$$

D) Any Taylor series can be written as:  $P(x) = P_n(x) + R_n(x)$

E)  $R_n(x)$  can be estimated as  $\frac{f^{(n+1)}(c)}{(n+1)!} (x-a)^{n+1}$  for some  $c$  between  $a$  and  $x$  for

the series centered at  $x = a$ . Its absolute value is referred to as the Lagrange error.

F) At  $x = b$ ,  $|f(b) - P_n(b)|$  is the truncation error.

G) To determine the radius and interval of convergence of an infinite series,

$$\text{Set } \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1 \text{ and solve. We must check the endpoints by}$$

plugging them in.

H) *The Alternating Series Test*: The series  $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$

converges if all three of the following conditions are satisfied:

1. each  $u_n$  is positive;
2.  $u_n \geq u_{n+1}$  for all  $n \geq N$ , for some integer  $N$ ;
3.  $\lim_{n \rightarrow \infty} u_n = 0$

D) *The Alternating Series Estimation Theorem*: If an alternating series satisfies the conditions listed above, then not only does it converge, but the truncation error is bounded by the next unused term.

## 17. CALCULATOR INFORMATION

A) A graphing calculator appropriate for use on the examinations is expected to have the built-in capability to:

1. plot the graph of a function within an arbitrary viewing window,
2. find the zeros of functions (solve equations numerically),
3. numerically calculate the derivative of a function, and
4. numerically calculate the value of a definite integral.

B) You do not have to show any work for those aspects listed above. You do need to show work on the free response section for items not listed. As an example, finding a relative maximum.

C) Graphing calculators with a QWERTY keyboard are not allowed on the examination.